Kármán–Howarth Theorem for the Lagrangian averaged Navier-Stokes alpha model

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Abstract

The Kármán–Howarth theorem is derived for the Lagrangian averaged Navier-Stokes alpha (LANS$-\alpha$) model of turbulence. Thus, the LANS$-\alpha$ model's preservation of the fundamental transport structure of the Navier-Stokes equations also includes preservation of the transport relations for the velocity autocorrelation functions. This result implies that the alpha-filtering in the LANS$-\alpha$ model of turbulence does not suppress the intermittency of its solutions at separation distances large compared to alpha.

1 Introduction

Kármán-Howarth (KH) theorem

The dynamics of the two-point velocity autocorrelation functions in turbulence contains information complementary to the spectral description. These autocorrelation functions directly connect the concept of scale with the result of an actual flow measurement. However, the autocorrelation functions yield no information about the energy, or enstrophy, that is contained in their interval of separation. Instead, the third-order correlation functions do provide information about the *fluxes* of energy and enstrophy as functions of separation distance. Thus, the spectral and autocorrelation dynamics give complementary information about the same phenomenon. The KH theorem relates the time derivative of the two-point velocity autocorrelation functions to the divergences of the third-order correlation functions. Perhaps not surprisingly, because of its fundamental importance, the line of investigation that began in 1938 with the KH theorem is still being actively pursued.

The invariant theory of isotropic turbulence was introduced by von Kármán & Howarth (1938) and refined by Robertson (1940), who reviewed the Kármán–Howarth (KH) theorem in the light of classical tensor invariant theory. The physical importance of the KH theorem in turbulence modeling is undeniable. According to Monin & Yaglom (1975) (vol II, page 122) the KH theorem's dynamical equation for the two-point autocorrelation function of the fluid velocity, "plays a basic part in all subsequent studies in the theory of isotropic turbulence." A homogeneous (but not necessarily isotropic) version of this theorem was introduced by Monin (1959). This non-isotropic

variant of the KH theorem was discussed further by Monin & Yaglom (1975) (vol II, page 403), but there was a gap in the proof. Correct proofs were given independently by Frisch (1995) and Lindborg (1996). These proofs are reviewed in Hill (1997), who concentrates on the logical steps needed to eliminate pressure-velocity correlations in the KH theorem without assuming isotropy.

The isotropic Kármán–Howarth theorem was extended to magnetohydrodynamics (MHD) in Chandrasekhar (1951) and was recently analyzed further using Elsässer variables by Politano & Pouquet (1998). Extensions of the KH theorem and its corollary – the "$-4/5$ law" of Kolmogorov (1941b) – to include passively advected scalars such heat and chemical concentration (or buoyancy) were given by Yaglom (1949) and were later refined by Antonia, Ould-Rouis, Anselmet & Zhu (1997), who also compared with Yaglom's results for the case of stratified fluids. The effect of fluid helicity on the KH theorem was discovered in Chkhetiani (1997) and analyzed in terms of structure functions in Gomez, Politano & Pouquet (1998). Arad, L'vov & Procaccia (1997) extended the fundamental results in Robertson (1940) by considering projections of the fluid velocity autocorrelation dynamics onto other irreducible representations of the $SO(3)$ symmetry group, besides the scalars under rotation. Modern computational resources are frequently applied, such as in Fukayama et al. (2000) and references therein, for numerically studying intermittency in turbulence, which is represented by the anomalous scaling behavior of its higher order structure functions, relative to those in the KH theorem.

Navier-Stokes alpha model

Recently, a modification of the Navier-Stokes (NS) equations known as the Navier-Stokes alpha model was introduced and its solution behavior was compared with experimental data for turbulent flow in pipes and channels in Chen et al. (1998, 1999a, 1999b). These modified equations essentially filter the fluid motion that occurs below a certain length scale (denoted as alpha), which is a parameter in the model. Amongst other differences from traditional turbulence modeling approaches, this alpha-filtering approach differs from the large eddy simulation (LES) approach by preserving the basic transport structure of the exact NS equations. Direct numerical simulations (DNS) of this model for forced homogeneous turbulence were performed in Chen et al. (1999c). Homogeneous turbulence decay was also simulated numerically using this model in Mohseni et al. (2000). Both these simulation studies showed the NS$-\alpha$ model to be considerably more computable than the exact NS equations, while preserving essentially the same solution behavior as NS at length scales larger than alpha. A review of the basic properties of the Navier-Stokes alpha model and its early development is given in Foias, Holm and Titi (2001a). See also Foias, Holm and Titi (2001b) and Marsden and Shkoller (2001) for recent analytical and geometrical results for this model.

The alpha model's original derivation (without viscous dissipation) in Holm, Marsden and Ratiu (1998a, 1998b) was motivated by seeking a generalization to n dimensions of the integrable one-dimensional shallow water equation due to Camassa and Holm (1993). This derivation was based on the Lie algebraic and geometric properties of Hamilton's principle for fluid dynamics in the Eulerian description. However, motivated by its later derivation as a Lagrangian averaged fluid model in Holm (1999), Holm (2001), and by a related derivation in Marsden and Shkoller (2001), we shall henceforth refer to this model as the Lagrangian-averaged Navier-Stokes alpha model, or the LANS$-\alpha$ model.

Although several analytical and geometrical results are known for the LANS$-\alpha$ model, many

open questions still remain. An outstanding question concerns the effect of alpha on intermittency in turbulence. One supposes that filtering at scales smaller than alpha would suppress intermittency on those scales. However, what is the effect of alpha-filtering at small scales on intermittency in the larger scales? Proving the KH theorem for the LANS$-\alpha$ model is a *required* first step toward addressing this outstanding question. Remarkably, the effects of the alpha-filtering are confined to separations of order $O(\alpha)$ or smaller, in the analogs of the KH theorem and in Kolmogorov's energy balance law proved here for the LANS$-\alpha$ model.

The KH theorem for the LANS$-\alpha$ model proved here provides an exact result that demonstrates how the introduction of the length scale alpha affects the dynamics of the velocity autocorrelations as a function of separation between two points fixed in the domain of an isotropic LANS$-\alpha$ flow. These effects turn out to be negligible at separations that are large compared to the filtering scale alpha ($r \gg \alpha$). To the extent that the same result persists in the higher correlation functions and structure functions for the LANS$-\alpha$ model, one may expect the intermittency of the alpha model to be the same as that for Navier-Stokes at large separations ($r \gg \alpha$).

The LANS$-\alpha$ model equations

The motion equation for the unforced LANS$-\alpha$ model is given by

$$\partial_t \mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} + v_j \nabla u^j + \nabla\Big(p - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2 \Big) = \nu\Delta\mathbf{v}\,,$$
$$\text{where}\quad \mathbf{v} \equiv \mathbf{u} - \alpha^2\Delta\mathbf{u}\,,\quad \nabla\cdot\mathbf{u} = 0\,.$$

The left side of this motion equation preserves the transport structure of ideal fluids, especially the Kelvin-Noether theorem, see Holm, Marsden and Ratiu (1998a, 1998b). Here, the specific momentum $\mathbf{v}$ is related to the fluid velocity $\mathbf{u}$, via the Helmholtz operator $1-\alpha^2\Delta$ which contains the length scale α. Although derived from a different approach, these equations are formally similar to the equations for the motion of a 2^{nd} grade fluid, but with a different dissipation operator. Following Camassa & Holm (1993) for the original one-dimensional version of this equation, the motion equation for the alpha model may also be expressed equivalently in a way that emphasizes its nonlocality, as

$$(1-\alpha^2\Delta)\Big(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u}\Big) + \nabla p = -\,\mathrm{div}\tau\,, \tag{1.1}$$
$$\text{where}\quad \tau \equiv \nabla\mathbf{u}\cdot\nabla\mathbf{u} + \nabla\mathbf{u}\cdot\nabla\mathbf{u}^T - \nabla\mathbf{u}^T\cdot\nabla\mathbf{u}\,,\quad \nabla\cdot\mathbf{u} = 0\,.$$

Similar forms of the motion equation for a 2^{nd} grade fluid also appear in Dunn & Fosdick (1974), modulo a crucial difference in dissipation.[1] Upon inverting the Helmholtz operator in three

[1]The formal similarity of the LANS$-\alpha$ model with the 2^{nd} grade fluid equation (8.1) on page 241 of Dunn & Fosdick (1974) resonates with a remark made already in von Kármán & Howarth (1938) page 197, "The expression ... for the correlation tensor is exactly of the same form as that for the stress tensor for a continuous medium when there is spherical symmetry." Mathematically, the crucial difference between the LANS$-\alpha$, and 2^{nd} grade fluid models lies in their different dissipation operators. In the motion equation (1.1) for the fluid velocity, $\mathbf{u}$, the LANS$-\alpha$ model has diffusive viscous dissipation $-\nu\Delta\mathbf{u}$, introduced in Chen et al. (1998, 1999a, 1999b) for their comparisons with turbulent flow data, while the 2^{nd} grade fluid has only damping. That is, $-\nu\Delta\mathbf{u}$ in equation (1.1) is replaced by $\mu\mathbf{u}$ in the motion equation for 2^{nd} grade fluids, for a positive constant μ. Physically, there are also significant differences in the interpretations of both the solutions and the parameters that appear in the equations for the two models.

dimensional Cartesian components, this becomes

$$\partial_t u_i + \partial_k\big(u_i u^k + \tilde{p}\,\delta_i^k \big) - \nu\Delta u_i = -\,\alpha^2\,\partial_k\left(g_\alpha * \tau_i^k \right),\ \text{where}\, i,j,k \in 1,2,3,$$
$$g_\alpha * \tau_i^k \equiv \int g_\alpha(|\mathbf{x}' - \mathbf{x}|)\,\tau_i^k(\mathbf{x}')\,d^3x'\,,\ \tau_i^k \equiv \left(u_{i,j}u^{j,k} + u_{i,j}u^{k,j} - u_{j,i}u^{j,k} \right).$$

Here $(1-\alpha^2\Delta)\tilde{p} = p$ and g_α is the Green's function for the Helmholtz operator in the flow domain with boundary conditions compatible with *satisfying NS on the boundary.* The free space Green's function for the Helmholtz operator in three dimensions is $g_\alpha(r) = r^{-1}\exp-(r/\alpha)$, which is the well-known Yukawa potential for $r = |\mathbf{x}' - \mathbf{x}|$. This free space Green's function should apply to the extent that the turbulence is isotropic (so that α can be taken as constant) and away from boundaries at distances greater than order $O(\alpha)$.

This form of the LANS$-\alpha$ equation demonstrates that the nonlinear stresses proportional to α^2 are filtered by the Green's function $g_\alpha(r)$ of the Helmholtz operator, whose filter width is of order $O(\alpha)$. Given α, the term $g_\alpha * \tau_i^k$ applies at separations r such that $\alpha/r \le 1$ and is negligible for $\alpha/r \ll 1$. For those larger separations, the LANS$-\alpha$ equation reverts to NS. The alpha-filtering is a regularization. With it, the solutions of LANS$-\alpha$ are well-posed and have a *finite dimensional global attractor* under L_2-bounded forcing, as shown in Foias, Holm and Titi (2001b). See also Marsden and Shkoller (2001) for a proof of their well-posedness in the presence of boundaries. Such well-posedness results are not known to hold for the NS equations. However, solutions of the LANS$-\alpha$ equations do converge to solutions of the NS equations uniformly as $\alpha \to 0$ for any positive viscosity, as also shown in Foias, Holm and Titi (2001b).

The equations governing the correlations

In deriving the correlation dynamics, one may regard $(\, g_\alpha * \tau'^k_j \,)$ as a subgrid scale (actually, sub$-\alpha$ scale) stress tensor arising from the alpha-filtering procedure represented by convolution with the Green's function $g_\alpha(r)$, which decreases exponentially in separation with scale length α. See Holm (1999), Holm (2001) and Marsden and Shkoller (2001) for further analysis and explanation of how this viewpoint arises, upon applying Lagrangian averaging. We denote $\mathbf{u}(\mathbf{x}', t) = \mathbf{u}'$ and begin our investigation of the correlation dynamics by computing the ingredients of the partial time derivative $\partial_t(v_i u'_j)$,

$$\partial_t v_i + \partial_k\big(v_i u^k + p\delta_i^k \ - \alpha^2\, u_{i,m}u^{m,k} \big) = \nu\Delta v_i\,,$$
$$\partial_t u'_j + \partial'_k\big(u'_j u'^k + \tilde{p}'\,\delta_j^k \big) + \alpha^2\,\partial_k\left(g * \tau'^k_j \right) = \nu\Delta' u'_j\,.$$

We cross multiply and add these equations, average the result $\overline{(\,\cdot\,)}$ and use statistical homogeneity in the following form, with $\xi \equiv \mathbf{x}' - \mathbf{x}$, in the traditional KH notation,

$$\frac{\partial}{\partial \xi^k}\,\overline{(\,\cdot\,)} = \frac{\partial}{\partial x'^k}\,\overline{(\,\cdot\,)} = -\,\frac{\partial}{\partial x^k}\,\overline{(\,\cdot\,)}\,,$$

to find

$$\partial_t\,\overline{(\,v_i u'_j\,)} \quad - \quad \frac{\partial}{\partial \xi^k}\,\overline{\big((v_i u^k \ - \ \alpha^2\,u_{i,m}u^{m,k})u'_j\,\big)} + \frac{\partial}{\partial \xi^k}\,\big(\,\overline{(v_i\,\tilde{p}')}\,\delta_j^k - \overline{(u'_j\,p)}\,\delta_i^k\,\big)$$
$$+ \quad \frac{\partial}{\partial \xi^k}\,\overline{\big(\,v_i\,(u'_j u'^k \ + \ \alpha^2\,g * \tau'^k_j\,)\big)} = 2\nu\Delta_\xi\overline{\big(\,(v_i u'_j\,)}\,,$$

where Δ_ξ is the Laplacian operator in the separation coordinate ξ. Next, we symmetrize in i,j and use the relation obtained from homogeneity,

$$\overline{(\,v_i u'_j u'^k + v_j u'_i u'^k\,)} = -\,\overline{(\,v'_i u_j u^k + v'_j u_i u^k\,)}\,,$$

to find the homogeneous correlation dynamics for LANS−α,

$$\partial_t\,\overline{(\,v_i u'_j + v_j u'_i\,)} - \frac{\partial}{\partial \xi^k}\Big(\mathcal{T}^k_{ij} - \alpha^2\,\mathcal{S}^k_{ij} - \Pi^k_{ij}\Big) = 2\nu\Delta_\xi\overline{(\,v_i u'_j + v_j u'_i)}\,. \tag{1.2}$$

In this equation, the symmetric tensors $\mathcal{T}^k_{ij}$, Π^k_{ij} and $\mathcal{S}^k_{ij}$ are defined as

$$\begin{aligned}
\Pi^k_{ij} &\equiv \overline{(\,v_i\,\tilde p'\,)}\,\delta^k_j + \overline{(\,v_j\,\tilde p'\,)}\,\delta^k_i - \overline{(\,u'_j\,p\,)}\,\delta^k_i - \overline{(\,u'_i\,p\,)}\,\delta^k_j\,,\\
\mathcal{T}^k_{ij} &\equiv \overline{(\,v_i u'_j + v_j u'_i + v'_i u_j + v'_j u_i\,)\,u^k}\,,\\
\mathcal{S}^k_{ij} &\equiv \overline{(\,u_{i,m} u'_j + u_{j,m} u'_i\,)u^{m,k}} + \overline{(\,v_i\,g*\tau'^k_j + v_j\,g*\tau'^k_i\,)}\,.
\end{aligned}$$

Imposing isotropy

We now suppose the LANS−α solution is isotropic and follow the approach of Kármán & Howarth (1938), as refined by Robertson (1940) and Chandrasekhar (1951) using the invariant theory of isotropic tensors. By a standard argument, isotropy implies we may drop the pressure-velocity tensor Π^k_{ij}. Hence, we rewrite equation (1.2) as

$$\frac{\partial}{\partial t}\,\mathcal{Q}_{ij} = \frac{\partial}{\partial \xi^k}\Big(\mathcal{T}^k_{ij} - \alpha^2\,\mathcal{S}^k_{ij}\Big) + 2\nu\Delta_\xi\,\mathcal{Q}_{ij}\,, \tag{1.3}$$

with the corresponding definition, $\mathcal{Q}_{ij} \equiv \overline{(\,v_i u'_j + v_j u'_i\,)}$. According to their definitions, the tensors $\mathcal{Q}_{ij}$ and $\mathcal{T}^k_{ij}$ are both symmetric and divergence-free in their indices i,j for constant α. For consistency with the isotropy assumption, equation (1.3) implies that $\mathcal{S}^k_{ij}$ must *also* be symmetric and divergence-free in its indices i,j.

Remark. Thus, DNS could check whether $\alpha \neq 0$ is consistent with isotropy in a given flow regime by checking whether the divergences of $\mathcal{S}^k_{ij}$ vanish in its indices i,j.

According to the classical theory of invariants discussed in Robertson (1940) and Chandrasekhar (1951), these three symmetric, divergence-free, isotropic tensors may each be expressed in terms of a single defining function. In particular, the isotropic $u-v$ autocorrelation tensor is given by

$$\mathcal{Q}_{ij} = \mathrm{curl}\,\big(Q\,\varepsilon_{ij\ell}\,\xi^\ell\,\big) = rQ'\Big(\frac{\xi_i\xi_j}{r^2} - \delta_{ij}\Big) - 2Q\delta_{ij}\,,$$

with defining function $Q(r,t)$ and $Q' = \partial Q/\partial r$ in the KH notation. The isotropic triple correlation tensor is

$$\begin{aligned}
\mathcal{T}^k_{ij} &= \mathrm{curl}\Big(T\,\big(\xi_i\varepsilon_{jk\ell}\,\xi^\ell + \xi_j\varepsilon_{ik\ell}\,\xi^\ell\,\big)\Big)\\
&= \frac{2}{r}T'\xi_i\xi_j\xi_k - (rT' + 3T)\Big(\xi_i\delta_{jk} + \xi_j\delta_{ik}\Big) + 2T\delta_{ij}\xi_k\,,
\end{aligned}$$

with defining function $T(r,t)$ and antisymmetric tensor $\varepsilon_{ij\ell}$. Hence, we compute the divergence,

$$\frac{\partial}{\partial\xi^k}\,\mathcal{T}^k_{ij} = \text{curl}\Big((rT' + 5T)\varepsilon_{ij\ell}\,\xi^\ell\Big) = \text{curl}\Big(\frac{1}{r^4}(r^5T)'\varepsilon_{ij\ell}\,\xi^\ell\Big)\,.$$

This is formula (45) in Chandrasekhar (1951) and, of course, it agrees with the corresponding formula (4.13) in Robertson (1940). Likewise, the isotropic mean sub$-\alpha$ scale stress tensor $\mathcal{S}^k_{ij}$ must also take the same form,

$$\frac{\partial}{\partial\xi^k}\,\mathcal{S}^k_{ij} = \text{curl}\Big((rS' + 5S)\varepsilon_{ij\ell}\,\xi^\ell\Big) = \text{curl}\Big(\frac{1}{r^4}(r^5S)'\varepsilon_{ij\ell}\,\xi^\ell\Big)\,,$$

with defining function $S(r,t)$.

Remark. Because to the presence of curl in their definitions, the defining functions T and α^2S have dimensions of energy dissipation rate, $(\overline{u^2})^{3/2}/r$. Their relations to the defining functions in Kármán & Howarth (1938) will be discussed in a later section.

The equations governing the defining scalars

According to Robertson (1940), the scalar defining the Laplacian of a second order isotropic tensor is obtained by operating with

$$D = \Big(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\Big) = \frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial}{\partial r}\,,$$

on the scalar defining the original tensor. That is,

$$D(Q) = r^{-4}(r^4Q')'\,.$$

The scalars defining the various second order tensors in equation (1.3) are, therefore,

$$\frac{\partial Q}{\partial t}\,,\quad \Big(r\frac{\partial}{\partial r}+5\Big)T = \frac{1}{r^4}(r^5T)'\,,\quad \Big(r\frac{\partial}{\partial r}+5\Big)S = \frac{1}{r^4}(r^5S)'\,,\quad D(Q)\,.$$

As Robertson (1940) points out, upon assuming isotropy, the tensor equation (1.3) is entirely equivalent to the corresponding scalar equation. This observation proves the following:

Theorem 1.1 (KH theorem for the LANS$-\alpha$ model) *Let the LANS$-\alpha$ model flow be isotropic. Then, the scalar equation*

$$\frac{\partial Q}{\partial t} = \Big(r\frac{\partial}{\partial r}+5\Big)(T-\alpha^2S) + 2\nu D(Q)\,, \tag{1.4}$$

defines an exact relation among the 2nd and 3rd correlation functions, and the stress tensor $\mathcal{S}^k_{ij}$.

Remark. Formula (1.4) is the analog for the LANS$-\alpha$ fluid equations of the KH theorem for NS turbulence.

Upon setting $\partial Q/\partial t = -2\,\overline{\varepsilon_\alpha}/3$ for the energy dissipation rate in three dimensions and dropping the viscous terms in equation (1.4) – as is appropriate for separation r in the inertial range – one finds the energy balance relation for the LANS$-\alpha$ model,

$$-\frac{2}{3}\,\overline{\varepsilon_\alpha} = \frac{1}{r^4}\frac{\partial}{\partial r}\Big(r^5(T-\alpha^2S)\Big)\,. \tag{1.5}$$

Here, $\overline{\varepsilon_\alpha}$ denotes the average dissipation rate of the total kinetic energy for the LANS−α model, given by $E_\alpha = \frac{1}{2}\int \mathbf{u}\cdot\mathbf{v}\, d^3x$. Integration of the energy balance relation (1.5) then proves the following:

Corollary 1.2 ("–2/15 law" for the LANS−α model) *In the inertial range and for arbitrary ratio α/r, the LANS−α model satisfies the "–2/15 law,"*

$$-\frac{2}{15}\,\overline{\varepsilon_\alpha} = T - \alpha^2 S\,. \tag{1.6}$$

Remark. Formula (1.6) is the analog for the LANS−α fluid equations of Kolmogorov's "–4/5 law" for NS turbulence.

Comparisons with NS turbulence theory for $\alpha/r \to 0$

Foias, Holm and Titi (2001b) show that solutions of the LANS−α model converge to solutions of the NS equations as $\alpha \to 0$ uniformly for any positive viscosity. Therefore, to compare the KH−α theorem with the results of Kármán & Howarth (1938) for the NS equations, we may consider the limit as $\alpha/r \to 0$. In this limit, one may neglect $\alpha^2 S$ in equations (1.5) and (1.6). We follow Robertson (1940) in identifying the KH double correlation scalars $f(r,t)$, $g(r,t)$ as

$$(\overline{u^2})f = Q\,, \quad \text{and} \quad (\overline{u^2})g = \frac{1}{2}rQ'\,. \tag{1.7}$$

Likewise, the KH triple correlation scalars h, k and q are identified in terms of $T(r,t)$, as $(\overline{u^2})^{3/2}h = rT/2$ for h, as well as,

$$(\overline{u^2})^{3/2}k = -\frac{1}{r^4}\int_0^r s^4 T\, ds \quad \text{and} \quad (\overline{u^2})^{3/2}q = \frac{1}{8r^4}\int_0^r s^4 T\, ds \,-\, \frac{r}{4}T\,,$$

for k and q. Using the relation $T = 2(\overline{u^2})^{3/2}h/r$ yields,

$$\Big(r\frac{\partial}{\partial r} + 5\Big)\,T = 2(\overline{u^2})^{3/2}\Big(\frac{\partial}{\partial r} + \frac{4}{r}\Big)\,h\,. \tag{1.8}$$

Hence, when $\alpha/r \to 0$, equation (1.4) of the KH−α theorem 1.1 yields equation (51) of Kármán & Howarth (1938), namely, the KH equation,

$$(\overline{u^2})\,\frac{\partial f}{\partial t} = \Big(\frac{\partial}{\partial r} + \frac{4}{r}\Big)\Big[2(\overline{u^2})^{3/2}\,h + 2\nu\,(\,\overline{u^2}\,)\,\frac{\partial f}{\partial r}\Big]\,. \tag{1.9}$$

One may refer to Monin & Yaglom (1975) page 122, for the KH equation in their notation. When the factor 1/6 relating third order autocorrelation functions and structure functions is introduced, this is also equation (3) of Kolmogorov (1941b), leading to the Kolmogorov's −4/5 law for Navier-Stokes fluids. See also Landau & Lifschitz (1987) for additional discussion of this fundamental result.

Thus, the limit $\alpha/r \to 0$ of formula (1.4) of the KH−α theorem recovers the expected classical results for homogeneous, isotropic, NS turbulence.

2 Conclusions

Main results. The main results of the paper are, as follows.

- Equation (1.2) for the LANS$-\alpha$ dynamics of the velocity autocorrelation functions with homogeneous statistics shows that the alpha model's preservation of transport structure extends to preserving the form of these transport equations, as well, modulo a shift in the term involving third moments that accommodates the contribution from the sub-alpha scale stress tensor.

- The modifications of the KH theorem and Kolmogorov's energy balance law for NS to include the effects of alpha-filtering in the LANS$-\alpha$ model are relatively simple and straightforward.

  - Equation (1.4) in the KH$-\alpha$ theorem gives the alpha-analog of the Karman-Howarth theorem for isotropic turbulence.

  - As a corollary of the KH$-\alpha$ theorem, equation (1.6) gives the "$-2/15$ law" alpha-analog of Kolmogorov's "$-4/5$ law" for energy balance in the inertial range. (The relative 1/6 in these coefficients arises from the 1/6 relation between autocorrelation functions and structure functions in isotropic turbulence.)

  - The second term in the $-2/15$ law in equation (1.6), the $\mathcal{S}-$term which is proportional to α^2, is very reminiscent of the quantity that appears in the corresponding -2 law for enstrophy cascade in 2D turbulence. The latter expression contains two powers of vorticity and one power of velocity. For example, see the Appendix B of Eyink (1996), where the identity is derived in detail. In the same way, the $\mathcal{S}-$term in (1.4) contains two powers of velocity gradient and one of velocity. It should thus be the dominant term (compared to the first $\mathcal{T}-$term) when $r < \alpha$ and the velocity is nearly smooth. It then corresponds to an"enstrophy-like" cascade, in agreement with the considerations of Foias, Holm and Titi (2001a) that determine the k^{-3} spectrum in that range.[2]

These exact results confirm that alpha-filtering affects the homogeneous isotropic statistical properties of NS only at separations r of order $O(\alpha)$, or smaller. This is because of the exponential decrease with separation in the Green's function for the Helmholtz operator. The same exponential decrease with separation applies for the dynamics of the higher order autocorrelation functions and structure functions, whose anomalous scaling properties for NS indicate the intermittency of turbulence. Consequently, one may expect the intermittency of the alpha model will be the *same* as for NS at large separations, for which $r/\alpha \gg 1$. This conclusion may apply more generally for other types of filtering, as well.

Some aspects of these results were not entirely unexpected. After all, one could enforce homogeneity and isotropy on the velocity correlation functions for *any* fluid model (e.g., an LES model, or the 2^{nd} grade fluid model). One could then compute its corresponding KH equation by the same classical methods as those used here. In general, this procedure could introduce complications, such as requiring several more scalar defining functions. Remarkably, the KH$-\alpha$ theorem for the case of constant alpha may be written at the cost of just one additional scalar function, S(r,t), arising from the sub-alpha scale stresses. The LANS$-\alpha$ model preserves the fundamental transport structure of the NS equations. It also preserves the form (1.3) of the

[2] I am grateful to G. Eyink for this observation.

transport relations for the velocity autocorrelation functions. Thus, the interpretations of its consequences turned out to be relatively straightforward in this case.

Helical turbulence. These results readily extend to helical turbulence, and are expected also to extend to any desired higher order autocorrelation equations. For example, one may expect that the fundamental relations for the third moments in helical turbulence obtained in Chkhetiani (1996) will also be expressed simply and concisely in the LANS$-\alpha$ model formulation. This will be discussed elsewhere.

Spectral representation. The spectral representation of these correlation dynamics is also worth examining further, especially because the inverse of the Helmholtz operator is algebraic in spectral space. As mentioned earlier, this direction was partially developed in Foias, Holm and Titi (2001a), where the $k^{-5/3} \to k^{-3}$ roll-off in the LANS$-\alpha$ energy spectrum was discovered in the spectral range for which $k\alpha < 1$ passes to $k\alpha > 1$. However, much remains to be studied about how the alpha-filtering in the LANS$-\alpha$ model affects (or preserves) other fundamental aspects of turbulence modeling, while at the same time making the resulting regularized turbulence model more computationally accessible.

Loitsyanski invariant. On the basis of our earlier reasoning that the LANS$-\alpha$ model results should recover the NS results as $\alpha/r \to 0$, we expect that the LANS$-\alpha$ predictions should agree with NS theory at large separation, particularly in regard to the behavior of the Loitsyanki integral.

Future directions.[3] The KH$-\alpha$ theorem confirms analytically that alpha-filtering leaves the velocity autocorrelation statistics invariant at sufficiently large separations $\alpha/r \ll 1$ (and, thus, could be expected to preserve intermittency at those separations) for homogeneous isotropic turbulence. This has three promising implications for future research directions:

(1) The alpha $\to \infty$ limit equation could be used to examine the small-separation effects of alpha on the statistics.

(2) One may study the dependence on the order of the correlation function of the transition from the α-dominated to the α-irrelevant regime. This would determine the α-dependence of the scaling regions.

(3) Needing only to compute scales significantly larger than a fixed value of alpha in LANS$-\alpha$ should provide more dynamic range for DNS than in the NS case, when alpha vanishes. This suggests one might fix alpha in the LANS$-\alpha$ model and use DNS to seek evidence of *universality at large scales*, especially in turbulence decay.

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